Research Project (Progetto di Ricerca)

Borel-definable refinements of classical algebraic invariants in topology have been recently introduced by Bergfalk, Lupini, and Panagiotopoulos in the context of Borel-definable homological algebra. This framework builds on initial intuitions of Moore and Beilinson–Bernstein–Deligne of replacing abelian groups as target category of algebraic invariants with a canonical "abelian envelope" of the category of abelian Polish groups. Such a category is called the *left heart*, and was later constructed and characterized by Schneiders for arbitrary quasi-abelian categories. The application of methods from descriptive set theory gave new impetus to this approach, leading to an explicit description of the left heart of abelian Polish groups in the work of Lupini. In this description, the objects are *groups with a Polish cover*, namely groups explicitly presented as the quotient of an abelian Polish group (the Polish cover) by a Polish subgroup (whose topology makes the inclusion continuous, but generally not a topological embedding). The morphisms are group homomorphisms that are *Borel-definable*, in the sense that admit a lift to a Borel function between the Polish covers. Analogous descriptions hold for the left heart of a number of important categories of algebraic structures endowed with a topology, including locally compact abelian Polish groups and separable Banach spaces. In this project, the successful applicant will apply this framework to one of the following areas of mathematics, depending on their expertise and inclinations.

Homological algebra

Many important invariants in homological algebras are endowed with canonical Polish group topologies, and thus their derived functors can be seen as taking values in the category of groups with a Polish cover. For example, this applies to Hom for countable abelian groups, the corresponding derived functor being Ext. This is a particularly important invariant in homological algebra, as a number of other algebraic invariants in topology, coarse geometry, and operator algebras can be expressed in terms of Hom and Ext by way of so-called Universal Coefficient Theorems.

The applicant interested in this area will obtain a better understanding Ext and other such derived functors when regarded as taking values in the left heart of abelian Polish groups and related categories, building on previous work of Bergfalk–Lupini–Panagiotopoulos and Lupini. Particularly, the applicant will focus on the following problems:

1. Characterize the complexity of (the trivial subgroup of) $\mathrm{Ext}(C, A)$ for countable abelian groups C and A, extending the work of Lupini in the torsion and torsion-free cases. More generally, describe the fine structure of $\mathrm{Ext}(C, A)$ provided by the Solecki subgroups, which are the canonical chain of subgroups with a Polish cover that are least among those of a given complexity.

2. Extend the results from (1) to the case of locally compact (totally disconnected) abelian Polish groups, building on the work of Moskowitz, Fulp–Griffith, and Hoffmann–Spitzweck.

3. Consider more generally the case of locally compact R-modules for a given countable (or Polish) ring R.

4. Use the results above to characterize injective and projective objects in the category of locally compact R-modules and its left heart.

Algebraic topology

Many important algebraic invariants can be constructed as homology groups of suitable complexes of Polish abelian groups, and can thus be regarded as functors to the left heart of abelian Polish groups. This was shown in the case of Steenrod homology and Čech cohomology by Bergfalk–Lupini–Panagiotopoulos. In this context, Lupini has obtained Borel-definable refinements of the classical Universal Coefficiet Theorems of Eilenberg–MacLane relating homology and cohomology. The successful applicant interested in topology will apply these Borel-definable refinements of classical algebraic invariant to obtain new and stronger classification results for spaces and maps, including the following problems:

1. Apply definable homology and cohomology to obtain new classification results for solenoid, solenoidal manifolds, and other fractals, as well as for their complements inside spheres, mapping telescopes, and other spaces with complicated boundary.

2. Classify up to homotopy maps between the spaces as in (1). Classify the orbits of the homotopy automorphism group on the space of homotopy classes of maps.

3. Refine the correspondence between Čech cohomology and homotopy classes of maps in terms of the Solecki subgroups, building on the work of Bergfalk–Lupini–Panagiotopoulos characterizing the 0-th Solecki subgroup in terms of phantom maps.

Coarse geometry

Algebraic invariants play an important role not only in topology, but also in other areas of mathematics such as coarse geometry. In this context, objects are identified when they have the same shape when considered "from infinitely far away". Coarse analogues of algebraic invariants include coarse homology, coarse cohomology, and coarse K-homology. The applicant interested in coarse geometry and related areas will apply Borel-definable methods in this context, focusing on the following problems:

1. Show that coarse homology, coarse K-homology, and coarse cohomology of coarse spaces can be regarded as groups with a Polish cover.

2. Apply the corresponding Borel-definable refinements of such invariants to obtain new classification results for coarse spaces that are complicated "at infinity", and for maps between them up to coarse homotopy.

3. Isolate the complexity-theoretic content of the Baum–Connes conjecture for coarse spaces, in terms of the group with a Polish cover structure on coarse K-homology and on K-theory of the translation C*-algebra.

4. Build examples of coarse spaces that witness a strong failure of the Baum–Connes conjecture, due to the following complexity-theoretic obstruction: projections on the translation C*-algebras are not classifiable using as invariants orbits of actions of pro-countable abelian groups.

Operator algebras

The study in the context of operator algebras of algebraic invariant inspired by topology goes back to the work of Brown–Douglas–Fillmore on K-homology. Later on, a noncommutative generalizations of K-theory was also considered, and a common generalization of K-homology and K-theory was introduced by Kasparov through the KK-bifunctor. Much of the work in operator algebras in the last 30 years has focused on the problem computing such invariants in the case of canonical examples of C*-algebras arising from mathematical physics, dynamical systems, number theory, or discrete mathematics, and on the attempt to classify "well-behaved" C*-algebras by means of such invariants together with additional "measure-theoretic" information enclosed in their space of tracial states, in the context of the Elliott Classification Programme.

The successful applicant with inclinations towards operator algebras will study Borel-definable refinements of algebraic invariants for C*-algebras, building on the work of Lupini on K-homology, and focussing on the following problems:

1. Show that KK-groups of C*-algebras can be regarded as groups with a Polish cover.

2. Isolate the complexity-theoretic consequences of the Universal Coefficient Theorem for C*-algebras of Rosenberg and Schochet.

3. Construct examples of C*-algebras that fail to satisfy the Universal Coefficient Theorem for complexity-theoretic reasons, due to the fact that their extensions are not classifiable using orbits of actions of pro-countable abelian groups as invariants.

Geometric group theory and bounded cohomology

An explicit description of the left heart for separable Banach spaces was given by Lupini, Moraschini, and Sarti in terms of spaces with a Banach cover and linear maps that admit an "approximately linear" continuous lift. This description opens up the possibility of studying ℓ^1-homology of groups and spaces as functors with values in the left heart of separable Banach spaces. The successful applicant with interest in geometric group theory and the study of cohomological invariants in that context will collaborate with Lupini, Moraschini, and Sarti to advance this project, aiming at settling the following problems:

1. Show that ℓ^1-homology of groups as spaces is a finer invariant when regarded as functor with values in the heart of separable Banach spaces.

2. Compute the corresponding fine structure of ℓ^1-homology given by the Solecki subspaces.

3. Characterize the left heart of the category of dual spaces of separable Banach spaces, and w*-continuous linear maps. Show that bounded cohomology can be seen as a functor to this category.

4. Introduce the functor Ext for spaces with a Banach cover and use it to establish a Universal Coefficient Theorem relating bounded cohomology and ℓ^1-homology.

Work-plan (Piano di Attività)

The successful applicant will engage in the following activities:

1. Review of the literature on Borel-definable refinements of classical algebraic invariants.
2. Review of the literature on the chosen subject of applications (homological algebra, algebraic topology, coarse geometry, operator algebras, bounded cohomology).
3. Work on the relevant items of the research project as listed above, including collaborations with Lupini and other staff members at UNIBO as well as collaborators from other institutions. This phase will involve research trips to visit collaborators.
4. Present the result of the research at research seminars at the University of Bologna and other Universities, and at relevant conferences as invited or contributed speaker.
5. Write research papers on the results of the research, to be submitted to prestigious peer-reviewed journals.
6. Write a survey paper on the project's area of research and its interactions with other applications of Borel-definable methods, to be submitted to the Bulletin of Symbolic Logic or another scientific journal that publishes surveys and expository articles.